7.53

We are told that the same flux passes through every turn of both coils. Call this flux Φ. Then the total flux passing through the first coil is $\Phi_1 = \Phi N_1$, while the total flux passing through the second is $\Phi_2 = \Phi N_2$. If one of the currents, say I_1, is changing, then there is an induced emf in coil 2 and a back emf in coil 1. The induced emf in coil 2 is given by $\mathcal{E}_2 = -d\Phi_2/dt = -N_2 d\Phi/dt$. The back emf in coil 1 is given by $\mathcal{E}_1 = -d\Phi_1/dt = -N_1 d\Phi/dt$. We see that the ratio of the induced emf to the back emf is just N_2/N_1.

7.55

(a)

Start from the charge conservation law:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}. \tag{1}$$

Differentiate both sides with respect to time:

$$\frac{\partial}{\partial t}\nabla \cdot \vec{J} = \nabla \cdot \frac{\partial \vec{J}}{\partial t} = 0 = -\frac{\partial^2 \rho}{\partial t^2}. \tag{2}$$

This implies that ρ is a linear function of time:

$$\rho(\vec{r}, t) = \rho(\vec{r}, 0) + \dot{\rho}(\vec{r}, 0)t. \tag{3}$$

(b)

We want to check that the Biot-Savart expression for the magnetic field,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}') \times \hat{\zeta}}{\zeta^2}, \tag{4}$$

with $\vec{\zeta} = \vec{r} - \vec{r}'$, satisfies the modified Ampere's law:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{5}$$

Taking the curl of the Biot-Savart law gives

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\nabla \times (\vec{J} \times \hat{\zeta})}{\zeta^2} = \frac{\mu_0}{4\pi} \int \left[-(\vec{J} \cdot \nabla) \frac{\hat{\zeta}}{\zeta^2} + \vec{J} \left(\nabla \cdot \frac{\hat{\zeta}}{\zeta^2} \right) \right]. \tag{6}$$

From equation 1.100 in the text, we know that

$$\nabla \cdot \frac{\hat{\zeta}}{\zeta^2} = 4\pi\delta(\vec{\zeta}), \tag{7}$$

so (6) becomes

$$\nabla \times \vec{B} = -\frac{\mu_0}{4\pi} \int (\vec{J} \cdot \nabla) \frac{\hat{\zeta}}{\zeta^2} + \mu_0 \vec{J}. \tag{8}$$

Note that this calculation is essentially identical to that given in section 5.3.2 of the text. Here too, we can replace the ∇ operator on unprimed coordinates with the operator $-\nabla'$ on primed coordinates. We may then do an integration by parts on the remaining integral in (8). As in section 5.3.2, we get two terms, one of which is a total divergence which becomes a vanishing surface integral. Whereas in section 5.3.2, the second term from the integration by parts was also vanishing, here it is not. It is of the form

$$-\frac{\hat{\zeta}}{\zeta^2} \nabla' \cdot \vec{J} = \frac{\hat{\zeta}}{\zeta^2} \frac{\partial \rho}{dt}. \tag{9}$$

Plugging this into (8) gives

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int \rho \frac{\hat{\zeta}}{\zeta^2} + \mu_0 \vec{J} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}. \tag{10}$$

In the last step, we used that the electric field is given by Coulomb's law:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho \frac{\hat{\zeta}}{\zeta^2}. \tag{11}$$

7.60

(a)

We will check this one equation at a time. First look at equation (i) of 7.43:

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho_e. \tag{12}$$

We make the replacements

$$\begin{aligned} \vec{E} \to \vec{E}' &= \vec{E}\cos\alpha + c\vec{B}\sin\alpha, \\ \rho_e \to \rho_e' &= \rho_e \cos\alpha + \frac{1}{c}\rho_m \sin\alpha. \end{aligned} \tag{13}$$

Plugging these into (12) and rearranging gives

$$\cos\alpha \nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho_e \cos\alpha + \left(\frac{1}{\epsilon_0 c}\rho_m \sin\alpha - c\sin\alpha \nabla \cdot \vec{B} \right). \tag{14}$$

The terms in parentheses cancel by virtue of Maxwell equation (ii), and after canceling the $\cos\alpha$ from the remaining terms, we get back equation (12). Equation (ii) of 7.43 is verified in a very similar manner.

Now consider equation (iii) of 7.43:

$$\nabla \times \vec{E} = -\mu_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t}. \tag{15}$$

We need to make the following replacements:

$$\begin{aligned} \vec{E} \to \vec{E}' &= \vec{E}\cos\alpha + c\vec{B}\sin\alpha, \\ \vec{J}_m \to \vec{J}_m' &= \vec{J}_m \cos\alpha - c\vec{J}_e \sin\alpha, \\ \vec{B} \to \vec{B}' &= \vec{B}\cos\alpha - \frac{1}{c}\vec{E}\sin\alpha. \end{aligned} \tag{16}$$

Plugging these into (15) gives

$$\cos\alpha \nabla \times \vec{E} + c\sin\alpha \nabla \times \vec{B} = -\mu_0 \vec{J}_m \cos\alpha + \mu_0 c \vec{J}_e \sin\alpha - \cos\alpha \frac{\partial \vec{B}}{\partial t} + \frac{1}{c}\sin\alpha \frac{\partial \vec{E}}{\partial t}. \tag{17}$$

The terms multiplying $\sin\alpha$ cancel by virtue of Maxwell equation (iv), and after canceling the remaining $\cos\alpha$ everywhere, we get back equation (15). Verification of equation (iv) follows similarly.

(b)

Here is the force law:

$$\vec{F} = q_e \left(\vec{E} + \vec{v} \times \vec{B} \right) + q_m \left(\vec{B} - \frac{1}{c^2}\vec{v} \times \vec{B} \right). \tag{18}$$

We need to do the following transformation:

$$\begin{aligned}\vec{E}' &= \vec{E}\cos\alpha + c\vec{B}\sin\alpha,\\ \vec{B}' &= \vec{B}\cos\alpha - \frac{1}{c}\vec{E}\sin\alpha,\\ q_e' &= q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha,\\ q_m' &= q_m\cos\alpha - cq_e\sin\alpha.\end{aligned} \tag{19}$$

Plugging in and expanding, we find three types of terms: ones having coefficient $\sin^2\alpha$, ones with $\cos^2\alpha$ and ones with $\sin\alpha\cos\alpha$. The terms with $\sin\alpha\cos\alpha$ all cancel. Here are the remaining terms:

$$\begin{aligned}&q_e\left(\vec{E} + \vec{v}\times\vec{B}\right)\cos^2\alpha + q_m\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right)\sin^2\alpha\\ &+q_e\left(\vec{E} + \vec{v}\times\vec{B}\right)\sin^2\alpha + q_m\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right)\cos^2\alpha.\end{aligned} \tag{20}$$

These terms can be combined pairwise to reproduce the force law (18).