Homework 5 Solutions

6.23

(a)

The transcription $\vec{D} \to \vec{B}$, $\vec{E} \to \vec{H}$, $\vec{P} \to \mu_0 \vec{M}$, $\epsilon_0 \to \mu_0$ tells us that the determination of $\vec{H}$ inside a uniformly magnetized sphere of magnetization $\vec{M}$ will be exactly the same calculation as that of $\vec{E}$ for a uniformly polarized sphere of polarization $\vec{P}$. The latter problem was done for us in example 4.2, with the result that $\vec{E} = -\vec{P}/3\epsilon_0$. This tells us that for our current problem, $\vec{H} = -\vec{M}/3$. This then tells us the magnetic field inside the sphere:

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \frac{2}{3}\mu_0 \vec{M}. \tag{1}$$

This agrees with equation 6.16 in the book.

(b)

The electrostatic analog of this problem was done in example 4.7 with the result

$$\vec{E} = \frac{3}{\epsilon_r + 2}\vec{E}_0. \tag{2}$$

$\vec{E}_0$ is the background field, and $\epsilon_r \equiv \epsilon/\epsilon_0$. Inside media, $\vec{B} = \mu\vec{H}$ and $\epsilon\vec{E} = \vec{D}$, so it is clear that we also have the correspondence $\epsilon \to \mu$. We want to find $\vec{B}$ inside the sphere, so we should rewrite our answer for the electrostatic problem in terms of $\vec{D}$, which is the analog of $\vec{B}$:

$$\vec{D} = \epsilon\vec{E} = \frac{3\epsilon}{\epsilon_r + 2}\frac{1}{\epsilon_0}\vec{D}_0. \tag{3}$$

Applying our transcription rules then gives us the $\vec{B}$ field:

$$\vec{B} = \frac{3\mu_r}{\mu_r + 2}\vec{B}_0, \tag{4}$$

where we define $\mu_r \equiv \mu/\mu_0$.

(c)

We have essentially already done this problem. We know that the average electric field over a sphere with an arbitrary charge distribution and total dipole moment $\vec{p}$ (eqn 3.105),

$$\vec{E}_{ave} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{R^3}, \tag{5}$$

is the same as the field due to a uniformly polarized sphere (eqn 4.14):

$$\vec{E} = -\frac{1}{3\epsilon_0}\vec{P}. \tag{6}$$

This is because the total dipole moment is given by $\vec{p} = \frac{4\pi R^3}{3}\vec{P}$. By the transcription quoted in the problem, we expect that the average $\vec{H}$ field over the sphere is the same as the field due to a uniformly magnetized sphere. We have found the latter in part (a) using the correspondence between electrostatics and magnetostatics:

$$\vec{H} = -\frac{1}{3}\vec{M} = \vec{H}_{ave}. \tag{7}$$

We can also show that a similar relation holds for the $\vec{B}$ field:

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \quad \Rightarrow \quad \vec{B}_{ave} = \mu_0(\vec{H}_{ave} + \vec{M}) = \frac{2}{3}\mu_0\vec{M}. \tag{8}$$

We can replace $\vec{M}$ with the total magnetic dipole moment of the sphere, $\vec{m} = \frac{4\pi R^3}{3}\vec{M}$, yielding

$$\vec{B}_{ave} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3}, \tag{9}$$

in agreement with equation 5.89.

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The electric field for a uniformly charged sphere of radius R and charge density ρ is given by

$$\vec{E} = \frac{\rho\hat{r}}{3\epsilon_0}\left\{ \begin{array}{ll} r & r < R \\ \frac{R^3}{r^2} & r > R \end{array} \right\}. \tag{10}$$

It is also given by an integral:

$$\vec{E} = \frac{\rho}{4\pi\epsilon_0}\int_{sphere} d^3r' \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{11}$$

From these two expressions we see that

$$\frac{1}{4\pi}\int_{sphere} d^3r' \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\hat{r}}{3}\left\{ \begin{array}{ll} r & r < R \\ \frac{R^3}{r^2} & r > R \end{array} \right\}. \tag{12}$$

First, we will compute the scalar potential of a uniformly polarized sphere using this result. The potential is

$$V = \frac{1}{4\pi\epsilon_0}\int_{sphere} d^3r' \frac{(\vec{r} - \vec{r}')\cdot\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\vec{P}}{4\pi\epsilon_0}\cdot\int_{sphere} d^3r' \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\vec{P}\cdot\hat{r}}{3\epsilon_0}\left\{ \begin{array}{ll} r & r < R \\ \frac{R^3}{r^2} & r > R \end{array} \right\}. \tag{13}$$

This agrees with the result given in example 4.2.

Next, we'll apply (11) to the computation of the vector potential for a uniformly magnetized sphere.

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\int_{sphere} d^3r' \frac{\vec{M}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\vec{M}\mu_0}{4\pi} \times \int_{sphere} d^3r' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \\ &= \frac{\mu_0 \vec{M} \times \hat{r}}{3} \left\{ \begin{array}{ll} r & r < R \\ \frac{R^3}{r^2} & r > R \end{array} \right\}. \end{aligned} \tag{14}$$

You can check that the curl of the expression for $r > R$ is what we found for the magnetic field of a uniformly magnetized sphere in part (a) of problem 6.23 above.