Homework 3 Solutions

5.50

(a)

We know the solution to the following set of equations:

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = \mu_0 \vec{J}. \tag{1}$$

This is just the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{2}$$

Therefore, if we want to solve this set of equations:

$$\nabla \cdot \vec{A} = 0, \qquad \nabla \times \vec{A} = \vec{B}, \tag{3}$$

it's obvious that we just need to take $\mu_0 \vec{J} \to \vec{B}$ in (2):

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \frac{\vec{B}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{4}$$

(b)

The solution to this set of equations:

$$\nabla \cdot \vec{E} = \rho/\epsilon_0, \qquad \nabla \times \vec{E} = 0, \tag{5}$$

is just

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{(\vec{r} - \vec{r}')\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{6}$$

We'd like to use an analogous set of equations to solve for the electric potential V in terms of $\vec{E}$. This is a bit trickier than in the magnetic case discussed in part (a) because V is not a vector, and we apparently only have one equation for V:

$$\nabla V = -\vec{E}, \tag{7}$$

which does not look very similar to the first equation of (5). In particular, it involves a different differential operator. There are several ways to get around this difficulty. Here, we will try to change (5) to make it look like (7). This is easily accomplished by taking the gradient of both sides of the first equation in (5):

$$\nabla(\nabla \cdot \vec{E}) = \frac{1}{\epsilon_0} \nabla \rho. \tag{8}$$

This equation is just like (7), with $V \to \nabla \cdot \vec{E}$ and $-\vec{E} \to \frac{1}{\epsilon_0} \nabla \rho$. The solution to (8) is of course just the divergence of (6):

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \nabla \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}, \tag{9}$$

where the divergence is on the unprimed coordinates. This divergence is worked out on page 50 of the text, and it is equal to 4π times a Dirac delta function, giving back equation (5) as it must. However, we don't want to do this simplification here; instead, we want to rewrite equation (9) in terms of $\nabla \rho$. First note that in (9) we may replace the divergence on unprimed coordinates $\nabla \cdot$ with minus the divergence on primed coordinates $-\nabla' \cdot$ because the thing being differentiated picks up a minus sign when we swap $\vec{r} \leftrightarrow \vec{r}'$. Then, using integration by parts, we get:

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \left[-\nabla' \cdot \left(\rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) + \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \cdot \nabla' \rho(\vec{r}') \right]. \tag{10}$$

The integral of the first term in the integrand becomes a surface integral at infinity when we apply the divergence theorem. We will assume that ρ vanishes sufficiently quickly as we go to infinity so that this surface integral vanishes. We are then left with

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \cdot \nabla' \rho(\vec{r}'). \tag{11}$$

We may now use the analogy between (7) and (8) to write down the potential in terms of $\vec{E}$:

$$V(\vec{r}) = -\frac{1}{4\pi} \int d^3r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \cdot \vec{E}(\vec{r}'). \tag{12}$$

5.53

(a)

The equations for $\vec{W}$ are exactly analogous to those of $\vec{A}$. In particular, $\vec{W}$ is related to $\vec{B}$ in exactly the same way that $\vec{A}$ is related to $\mu_0 \vec{J}$. We can therefore immediately write down an expression for $\vec{W}$ in terms of $\vec{B}$:

$$\vec{W}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \frac{\vec{B}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \tag{13}$$

(b)

Making use of the hint, we know that for a constant $\vec{B}$ field, we can write the vector potential in cylindrical coordinates as

$$\vec{A}(s) = \tfrac{1}{2} sB\hat{\phi}. \tag{14}$$

Here, we have chosen to orient the $\vec{B}$ field along the z direction. This expression is the same as that given in problem 5.24 with $\vec{B}$ taken to be in the z direction. (Note, however, that the expression in 5.24 is not unique.) We don't want to plug this into (13) because that expression for $\vec{W}$ is only valid for magnetic fields that behave appropriately at spatial infinity. Here, the magnetic field is constant everywhere, so we should look for another way to solve the problem. Fortunately, there is a lot of symmetry in the setup, so we can use the analog of Ampére's law:

$$\oint d\vec{\ell} \cdot \vec{W} = \int d\vec{a} \cdot \nabla \times \vec{W} = \int d\vec{a} \cdot \vec{A}. \tag{15}$$

Now the right hand rule that we use to determine the direction of a magnetic field from the direction of the current that generates it stems directly from Maxwell's equation. This same rule therefore describes the relationship between the directions of $\vec{A}$ and $\vec{B}$ as well as between those of $\vec{W}$ and $\vec{A}$. Here, the right hand rule tells us that $\vec{W}$ must be in the z direction. Therefore, we choose a rectangular loop with two sides of length L parallel to the z axis, one at $s = 0$ and one at $s = R$. The other two sides connect the first two and are perpendicular to the z axis, so their length is immaterial. We can also conclude that $\vec{W}$ depends only on s by the symmetry of the problem ($\vec{A}$ depends only on s, and $\vec{B}$ is independent of all coordinates). Using these facts, the integral becomes

$$\oint d\vec{\ell} \cdot \hat{z} W(s) = L\left(W(0) - W(R)\right) = \frac{1}{2} B \int d\vec{a} \cdot \hat{\phi} s = \frac{1}{4} R^2 BL. \tag{16}$$

In the above, we've chosen to orient the loop so that $d\vec{a}$ is in the positive $\hat{\phi}$ direction. $W(0)$ is just an arbitrary constant which we set to zero. Therefore, we find that

$$\vec{W}(s) = -\frac{1}{4} s^2 B\hat{z}. \tag{17}$$

I leave it to you to check that $\nabla \times \vec{W} = \vec{A}$ and $\nabla \cdot \vec{W} = 0$.

(c)

From example 5.12 in the book, the vector potential inside and outside an infinitely long solenoid is

$$\vec{A}(s) = \frac{\mu_0 n I}{2}\hat{\phi}\left\{\begin{array}{cc} s & s < R \\ \frac{R^2}{s} & s > R \end{array}\right\}. \tag{18}$$

Inside the solenoid, the magnetic field is constant and pointing in the z direction, so we have the same story as in part (b) above. For outside the solenoid, we can choose the same loop as in part (b) (except we'll take the two sides parallel to z to be at s_0 and s) and again do Ampere's law:

$$W(s) = -\frac{R^2 B}{2}\int_{s_0}^{s} ds' \frac{1}{s'} = -\frac{BR^2}{2}\log\frac{s}{s_0}. \tag{19}$$

Here, $B = \mu_0 n I$ is the magnitude of the magnetic field inside the solenoid. s_0 is arbitrary since adding a constant to $\vec{W}$ doesn't change $\vec{A}$. However, if we choose the $\vec{W}$ field inside the solenoid to be exactly the same as in part (b), then this arbitrariness is fixed already since we made the choice $\vec{W}(0) = 0$. s_0 is then fixed by demanding that $\vec{W}$ be continuous across the boundary of the two regions, ie at the solenoid:

$$W_{inside}(s = R) = -\frac{1}{4}BR^2 = W_{outside}(s = R) = -\frac{1}{2}BR^2\log\frac{R}{s_0} \Rightarrow s_0 = \frac{R}{\sqrt{e}}. \tag{20}$$

In summary,

$$\vec{W}(s) = -\tfrac{1}{2}B\hat{z}\left\{\begin{array}{cc} s^2/2 & s < R \\ R^2\log\left(s\sqrt{e}/R\right) & s > R \end{array}\right\}. \tag{21}$$

5.56

(a)

Denote the angular frequency by ω: the donut rotates through ω radians per unit time. The current is thus $I = Q\omega/2\pi$, and the magnetic moment is $m = I \times \pi R^2 = \frac{1}{2}QR^2\omega$, where R is the radius of the donut. The angular momentum of the donut is $L = MR^2\omega$. The gyromagnetic ratio is therefore $m/L = Q/2M$.

(b)

We will take Griffiths advice and think of the sphere as a collection of rings. We can solve the problem without further calculation beyond that of part (a) by making a series of observations. First, note that the total dipole moment will just be the sum of all the dipole moments of all the constituent rings. From the result of part (a), we see that we may write the dipole moment of each ring in terms of its angular moment: $dm = (dQ/2dM)dL$. Here, dm, dQ, dM, and dL refer to the dipole moment, charge, mass, and angular momentum of some particular ring in the sphere. Now, since the sphere has uniform charge and mass, the ratio dQ/dM must be the same for every ring, and furthermore, must be the same as the charge to mass ratio of the entire sphere, Q/M. Therefore, the dipole moment of each ring may be expressed in terms of its angular momentum according to $dm = (Q/2M)dL$. As we have said, the dipole moment of the entire sphere is just the sum of the moments of all the constituent rings. The same can be said for the angular momentum of the sphere, so we have $m = (Q/2M)L$. The gyromagnetic ratio is therefore $Q/2M$. Notice that we haven't used any details about the sphere, only that it could be decomposed into rings. Therefore, the same answer applies to any object that can be thus decomposed.

We can also show the result for the sphere by direct calculation. The charge density of the sphere is $\sigma = Q/4\pi R^2$. (Note: if you did the solid sphere here, you should receive full credit.) The charge of each ring is $dQ = \int_0^{2\pi} \sigma R^2 \sin\theta d\theta d\phi = 2\pi\sigma R^2 \sin\theta d\theta$. The current due to each ring is $dI = \omega dQ/2\pi = \omega\sigma R^2 \sin\theta d\theta$. The dipole moment per ring is $dm = \pi(R\sin\theta)^2 dI = \pi\omega\sigma R^4 \sin^3\theta d\theta$. Integrating this over θ gives the dipole moment of the sphere: $m = \omega Q R^2/3$.

Next, we compute the angular momentum of the sphere. The mass density is $\sigma_M = M/4\pi R^2$. The mass per ring is $dM = 2\pi\sigma_M R^2 \sin\theta d\theta$. The angular momentum per ring is then $dL = dM(R\sin\theta)^2\omega = 2\pi\omega\sigma_M R^4 \sin^3\theta d\theta$. Integrating this over θ gives $L = 2\omega M R^2/3$, so the gyromagnetic ratio is $Q/2M$ as claimed above.

(c)

In the case of the electron, we take $L = \hbar/2$. Multiplying this by the gyromagnetic ratio $e/2m_e$, we get $e\hbar/4m_e$ for the magnetic dipole moment. This is the correct answer up to the "g factor" of the electron, which is approximately 2.